

Simple representation of a large class of non-uniform cellular automata

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Abstract. We constructed a directed labeled graph with only 36 vertices that characterizes all number-conserving one-dimensional binary non-uniform cellular automata with the four-cell neighborhood, for all finite grids of length at least seven.

Keywords: non-uniform cellular automata · number conservation · finite grids.

Our research concerns non-uniform cellular automata (ν -CAs), *i.e.*, cellular automata for which different cells are allowed to follow different local rules. In particular, we investigate one-dimensional binary ν -CAs on a finite grid with the four-cell neighborhood, or in other words, with radius $3/2$.

For a positive integer n we consider a finite grid of n cells $\mathcal{G}_n = \{0, 1, \dots, n-1\}$. Any n -tuple $\mathbf{x} = (x_0, x_1, \dots, x_{n-1}) \in \{0, 1\}^n$ is called a configuration of length n and represents the states of the cells in $\mathcal{G}_n$. We consider periodic boundary conditions: the last cell $n-1$ is adjacent to the first cell 0.

The local rule of a ν -CA with the four-cell neighborhood is a function $f : \{0, 1\}^4 \rightarrow \{0, 1\}$. We denote the set of all such local rules by $\mathcal{F}_4$. Each rule sequence $[f_0, f_1, \dots, f_{n-1}] \in (\mathcal{F}_4)^n$ generates a ν -CA $H : \{0, 1\}^n \rightarrow \{0, 1\}^n$ defined by $H(\mathbf{x})_i = f_i(x_{i-1}, x_i, x_{i+1}, x_{i+2})$.

We study ν -CAs in the context of number conservation. For a configuration $\mathbf{x}$, let $\mu(\mathbf{x}) = \sum_{i \in \mathcal{G}_n} x_i$, *i.e.*, $\mu(\mathbf{x})$ is the sum of all states in $\mathbf{x}$. A ν -CA $H : \{0, 1\}^n \rightarrow \{0, 1\}^n$ is called number-conserving if $\mu(H(\mathbf{x})) = \mu(\mathbf{x})$ for every $\mathbf{x} \in \{0, 1\}^n$.

In recent years, the detailed classification of all number-conserving ν -CAs with radius one was provided in [1]. Interestingly, the behaviors observed there are not particularly complex: every number-conserving ν -CA with radius one is either one of the five uniform CAs or a concatenation of blocks belonging to just four families.

Our research on ν -CAs with radius $3/2$ indicates a much greater complexity than that for radius one. In particular, it is easy to find numerous examples of number-conserving ν -CAs that are neither uniform nor made of blocks.

In [2] we described how to generate the directed graph Π with the following property: for each $n \geq 7$ there is a one-to-one correspondence between the set of all number-conserving ν -CAs with radius $3/2$ on a finite grid $\mathcal{G}_n$ and the set of all length- n closed directed walks in Π .

The graph Π has 273 538 vertices and 2 301 044 edges. It contains four strongly connected components Π_1 , Π_2 , Π_3 and Π_4 with 136 768, 136 768, 1 and 1 vertices, respectively. Moreover, Π_2 is a reflection of Π_1 , so to obtain results on number-conserving ν -CAs it is sufficient to explore Π_1 .

In [3] we used the idea of barriers and blocks developed in [1, 2]. In particular, we proved that if a sequence of local rules $f_i, \dots, f_k$ of length $l \geq 2$ is a block in some number-conserving ν -CA, then it is a block in every number-conserving ν -CA it appears in. This means that blocks can be explored independently of ν -CAs. Additionally, we found out that the set $\mathcal{L}$ of local rules that can be a first rule of a block has the cardinality of 36.

The following definition is crucial to our recent results.

Definition 1. Let $B = |h, h_1, h_2, \dots, h_{n-1}|$ be a block. We will call every block $A = |f, g, h_1, h_2, \dots, h_{n-1}|$ a left expansion of B . Additionally, we will regard every block of length two as a left expansion of $|\text{Id}|$.

One can show that the values of f, g and h depend only on each other and not on the local rules $h_1, h_2, \dots, h_{n-1}$. For a given $h \in \mathcal{L}$ let us denote by $\mathcal{C}(h)$ the set of couples (f, g) such that for any block B a block A is a left expansion of B . We can prove the following.

Theorem 1. For every block $A = |f, g, h_1, h_2, \dots, h_{n-1}|$ of length $n + 1$ there exists an **unique** block $B = |h, h_1, h_2, \dots, h_{n-1}|$ of length n such that A is a left expansion of B .

This allows us to construct a directed graph $\Gamma_1 = (V_1, E_1)$ with the set of vertices $V_1 = \mathcal{L}$ and the set of labeled edges

$$E_1 = \{h \xrightarrow{g} f \mid h \in \mathcal{L} \wedge (f, g) \in \mathcal{C}(h)\}.$$

By definition $|V_1| = 36$ and it is easy to show that $|E_1| = 225$. Moreover, we can show that there exists a one-to-one correspondence between blocks and closed directed walks in Γ_1 that start and end in the vertex $\text{Id} \in \mathcal{L}$. Finally, we can prove the following crucial property of Γ_1 .

Theorem 2. For every $n \geq 7$ there exists a one-to-one correspondence between length- n closed directed walks in Π_1 and length- n closed directed walks in Γ_1 .

Note that this means that the exploration of all number-conserving ν -CAs with the four-cell neighborhood can be reduced to the exploration of Γ_1 , since Π_2 is just a reflection of Π_1 . Therefore for every number-conserving ν -CA H on a grid of length $n \geq 7$ (with the exception of two simple uniform cases) either H corresponds to a closed directed walk in Γ_1 or the reflection of H does.

We are hopeful that Γ_1 will open new avenues of research on number-conserving ν -CAs. Thanks to it we already obtained results on the total number of number-conserving ν -CAs of a given length, as well as the total number of blocks of a given length.

References

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